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STEADINESS IN THE FLOW OF A WELL-CONDUCTING FLUID ACROSS A MAGNETIC FIELD

(Ob ustoychivosti techeniya khorosho provodyashchey zhidkosti poperek magnitnogo polya)

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1. The flow of a conducting incompressible viscous fluid between plane-parallel plates across an exterior uniform field, frozen-in an ideal conductor, outside the fluid, has been investigated by J. Gartman [1]. The precise solution of the stationary problem obtained by him, depends on the dimensionless parameter

$$M = \frac{lH_0}{c}\sqrt{\frac{\sigma}{\eta}} \qquad (1)$$

(here l is the half-width of the channel, H_0 is the outer field, σ and η respectively are the conductivity and the dynamic viscosity of the fluid, c is the speed of light).

If $M \ll 1$, the flow becomes a Poiseuil flow. But if the velocity of the flow is constant along the channel's cross section everywhere except in two thin boudary layers of $\sim l/M$ thickness, in which the velocity drops to zero. The velocity magnitude is determined by the correlation

$$V_0 = -\frac{dP}{dx}\frac{lc}{H_0\sqrt{\sigma\eta}}, \qquad (2)$$

where dP/dx is a constant pressure gradient in the direction of motion.

The oriented along the velocity magnetic field H_x varies by cross section at $M \gg 1$ so :

$$H_x = H_0 \lambda z, \tag{3}$$

where the coordinate z is counted as of the symmetry plane, along H_0, and

$$\lambda = \frac{4\pi}{H_0^2} \frac{dP}{dx}$$

is the reciprocal length's dimension constant. The correlation (3) is only disturbed in the boundary layers, where H_x drops to zero.

Let us note, that if one passes into a system of coordinates moving with a velocity V_0, the examined boundary solution coincides with the solution of the problem of equilibrium of an ideally-conducting fluid at same geometry in the gravitational field

$$g = -\frac{1}{\rho} \frac{dP}{dx}.$$

The lines of force of the magnetic field in the fluid have the shape of a parabola with a sagging depth $L = l^2 \lambda/2$.

The effect of velocity profile variation on the appearance of the usual turbulence (vortex untwisting by the velocity gradient) was investigated by Lock [2]. The effect on the steadiness of the magnetic field configuration also offers interest in being examined.

2. Let us consider that $M \gg 1$, and $V_0 \ll c$. Then H_x is given by the correlation (3) in the system of coordinates moving with the fluid $H_z = H_0$. The boundary layers are approximated by surface currents, in connection with which we shall drop the requirement of transforming at boundaries the tangential components of velocity and field perturbations into zero. Let us also assume that the magnitudes ν and $c^2/4\pi\sigma$ are sufficiently small, and let us neglect the dissipation at unstationary processes.

Assume that a small perturbation of the velocity $\mathbf{v}$ and of the magnetic field $\mathbf{h}$ has appeared. Linearizing the magnetic hydrodynamics' equations, we shall obtain :

$$\frac{\partial \mathrm{h}}{\partial t} + v_z \frac{d\vec{\mathscr{H}}}{dz} = (\vec{\mathscr{H}}\nabla)\,\mathrm{v}, \tag{4}$$

$$\frac{\partial}{\partial t}\,\mathrm{rot}\,\mathrm{v} = \frac{1}{4\pi\rho}\,\mathrm{rot}\left\{(\vec{\mathscr{H}}\nabla)\,\mathrm{h} + h_z \frac{d\vec{\mathscr{H}}}{dz}\right\} \tag{5}$$

$\mathscr{H}_x = H_0\lambda z,\ \mathscr{H}_y = 0,\ \mathscr{H}_z = H_0)$.

Aside from (4), we have :

$$\mathrm{div}\,\mathrm{h} = 0, \quad \mathrm{div}\,\mathrm{v} = 0. \tag{6}$$

At $z = \pm l$, h_z and v_z are equal to zero. Hence, we may obtain with the help of (4) the conditions for v_z:

$$\frac{\partial v_z(\pm l)}{\partial z} = 0. \tag{7}$$

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Excluding $\mathbf{h}$ from the equations, we have

$$\frac{\partial^2}{\partial t^2}\,\mathrm{rot}\,\mathrm{v} = \frac{1}{4\pi\rho}\,\mathrm{rot}\,\{(\vec{\mathscr{H}}\nabla)\,\mathrm{v}\}. \tag{8}$$

Let us examine the perturbations lying in the plane xz and not depending upon y. We shall seek the solution in the form

$$\mathrm{v} = \mathrm{v}\,(z)\, e^{i(\omega t + kx)}. \tag{9}$$

Excluding v_x, we shall obtain the following equation for

$$\frac{d}{dz}\left\{\omega^2 + \frac{H_0^2}{4\pi\rho}\left(2ik\lambda z + \frac{d}{dz}\right)^2\right\}\frac{dv}{dz} = k^2\left\{\omega^2 + \frac{H_0^2}{4\pi\rho}\left(2ik\lambda z + \frac{d}{dz}\right)^2\right\} v. \tag{10}$$

Let us introduce the dimensionless coordinate $s = z/l$ and the parameters

$$\varkappa = l^2 k\lambda = \frac{4\pi}{H_0^2}\frac{dP}{dx} k l^2, \quad \Lambda = \frac{l\omega\sqrt{4\pi\rho}}{H_0}.$$

Equation (10) takes the form

$$\frac{d}{ds}\left\{\Lambda^2+\left(2i\varkappa s+\frac{d}{ds}\right)^2\right\}\frac{dv}{ds}=(kl)^2\left\{\Lambda^2+\left(2i\varkappa s+\frac{d}{ds}\right)^2\right\}v. \qquad (10')$$

The total investigation of the equation (10') is beset with computational difficulties. That is why we shall limit ourselves to the examination of one of the particular cases, representing a self-sustaining interest for certain problems of cosmic electrodynamics.

3. If we examine long waves $(kl \leqslant 1)$, we may neglect the right-hand part of equation (10'), since it is proportional to the small parameter at the junior derivative. Let us however consider the magnetic field so weak, that and that possible are long waves for which simultaneously (this requirement may be satisfied at great $v_z(\pm l) = 0$

Neglecting the right-hand part of (10'), and taking into account the operator's identity

$$\left(2i\varkappa s+\frac{d}{ds}\right)f=e^{-i\varkappa s^2}\frac{d}{ds}e^{i\varkappa s^2}f, \qquad (11)$$

we have

$$\frac{d}{ds}\left\{\Lambda^2+e^{-i\varkappa s^2}\frac{d^2}{ds^2}e^{i\varkappa s^2}\right\}\frac{dv}{dx}=0. \qquad (12)$$

One of the solutions, v = const, is obvious, while the remaining are found with the aid of quadratures. Two solutions are even, and two are odd :

$$v_1=1; \quad v_2=\int_0^s \sin\Lambda s' e^{-i\varkappa s'^2}ds';$$

$$v_3=\int_0^s \cos\Lambda s' e^{-i\varkappa s'^2}ds'; \quad v_4=\int_0^s\int_0^{s'} e^{-i\varkappa(s'^2-s''^2)}\sin\Lambda(s''-s')\,ds''\,ds'. \qquad (13)$$

For even solutions the characteristic equation is trivial : $\sin\Lambda = 0$. Hence $\Lambda_n = n\pi$. All the solution are steady.

After simple but cumbersome transformations, we obtain that in case of odd solutions :

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$$\frac{\operatorname{tg}\Lambda}{\Lambda}=\frac{1}{\varkappa\Lambda}\frac{\int\limits_0^1(\sin\varkappa\cos\varkappa x^2-\cos\varkappa\sin\varkappa x^2)\frac{\sin\Lambda\cos\Lambda x-x\cos\Lambda\sin\Lambda x}{1-x^2}dx}{\left(\int\limits_0^1\cos\varkappa x^2\cos\Lambda x\,dx\right)^2+\left(\int\limits_0^1\sin\varkappa x^2\cos\Lambda x\,dx\right)^2}. \quad (14)$$

In the right-hand part the function is bounded and that is why its diagram intersects all the branches of the $\operatorname{tg}\Lambda/\Lambda$ with the exception, perhaps, of the branch lying between $-\pi/2$ and $\pi/2$. Consequently there exists an infinite multiplicity of steady solutions. The unsteadiness may only be linked with the indicated branch.

Let us examine wavelengths $L \gg l$. Taking out of the integrals integrals slowly-varying functions, and substituting them by values in zero, let us integrate the fast-oscillating functions from 0 to ∞. Assuming $\gamma = i\Lambda$, we obtain

$$\operatorname{ch}\gamma=\frac{\sqrt{\pi\varkappa}}{\sin(\varkappa-\pi/4)}, \qquad (v_z(\pm l)=0,$$

from where (considering that γ is sufficiently great and ...

$$\gamma=\pm\ln\frac{\sin(\varkappa-\pi/4)}{2\sqrt{\varkappa\pi}}. \quad (16)$$

At $\varkappa\to\pi/4+n\pi$ the requirement of slow variation of Λ-containing terms is not fulfilled. However, it is clear, that at $\varkappa$ approaching $\pi/4+\pi n$, a resonance increment growth steps on. If $\sin(\varkappa-\pi/4)>0$, the unsteadiness has a character of stationary waves, but if $\sin(\varkappa-\pi/4)<0$, travelling waves are excited. The instability increases most intensively when a whole plus one quarter number of waves is packed over the length L.

The increment, expressed through dimensional magnitudes, has the form

$$\frac{1}{\tau}=\frac{H_0}{l\sqrt{4\pi\rho}}\ln\left\{\frac{H_0}{4\pi l\sqrt{k\,dP/dx}}\sin\left(\frac{4\pi}{H_0^2}\frac{dP}{dx}kl^2-\frac{\pi}{4}\right)\right\}. \quad (17)$$

It may be seen from (17) that γ and σ play no part, if besides $M \gg 1$ the inequality $lH_0 \gg \sqrt{4\pi\rho\nu}$ is fulfilled.

As the unsteadiness develops, the lines of force close up by themselves in some regions, while in others the transverse field is stenghtnend. It may be said, that the exterior field is dislodged, concentrating in thin layers. Such configuration is in its turn unsteady, while both types of unsteadiness, studied by Kruskal and Schwartzshield [4] take place simultaneously – that in the gravitational field and the flexible cable with current.

The development of these instabilities must lead to the formation of clusters with locked (closed) lines of force, capable of moving freely in space.

4. Above was examined a case, when the density of the potential, and consequently of the kinetic energy is much greater than that of the exterior field's magnetic energy. Phenomena, at which the initially weak field is strenghtened as a consequence of the expansion of lines of force during the motion of conducting masses, are linked with similar circumstances.

The unsteadiness investigated in the preceding paragraphs limits the categories of flows at which a magnetohydrodynamic self-excitation of regular magnetid fields is possible, for cases to be excluded are those, when the lines of force have in specific regions the shape of strongly elongated parabolas.

Let us pause briefly on the problem of the turbulent self-excitation of the field. Two different viewpoints prevail in this question : Batchelor [4] considers that turbulent pulsations disrupt the magnetic field correlation, thereby linking it with the minimum turbulence scale, and that is why it has a correspondingly low energy. Other authors (see for example [5, 6]) assume that the field correlation is preserved, and that in the state of dynamic equilibrium the energy of the field is of the order of that of the fundamental flow scale.

In connection with the above considerations, the Batchelor's viewpoint prevails, for there exists a real mechanism of diruption of magnetic field correlation. It seems probable that this mechanism is not unique and that a similar unsteadiness takes place in other cases at strong line of force deformations.

***** THE END *****

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